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TORSIONAL BUCKLING OF AN ELASTIC THICK-WALLED TUBE MADE OF RUBBER-LIKE MATERIAL

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Abstract—The elastic stability of a rubber-like, thick-walled tube which is subjected to finite torsional deformation is investigated both theoretically and experimentally. The analysis is based on the theory of finite elastic deformations, in cojunction with the method of small displacements superposed on large elastic deformations. The governing field equations are solved by a numerical scheme which determines the critical buckling torque and the associated buckling mode of the tube. The predicted results compare closely with the experimental measurements of the buckling of thick-walled silicone rubber tubes tested under finite twist.

INTRODUCTION

Problems dealing with the finite deformations of elastic solids have received considerable attention during the last two and a half decades. Solutions of problems having physical importance can be found in such classical texts as those of Green and Zerna[1], and Green and Adkins[2]. The methods employed in the solutions of these problems are essentially either an inverse or a semi-inverse procedure in which a class of kinematically permissible deformations is assumed for the problem concerned and then the surface tractions required to produce the assumed deformation are determined. These procedures, however, do not provide a criterion for the stability of equilibrium of the assumed deformation.

When the deformation of a thick or thick-walled elastic body is finite, the analysis of its stability of equilibrium is usually necessary. For bodies undergoing large elastic deformations, one of the commonly adopted, well-known methods of stability analysis is the theory of small displacements superposed on large deformations[3]. Wilkes[4] was probably the first to apply the theory given in [3] in the study of the symmetrical buckling of a thick-walled circular tube under uniform end thrust. Later, using the same approach, Green and Spencer[5] studied the stability of a solid circular cylinder subjected to finite extension and torsion. Wesolowski[6] investigated the stability of thick-walled spherical shells under uniform external pressure, and Nowinski and Shahinpoor[7] examined cylindrical shells under uniform external pressure. Because these last two problems were only partially treated, Wang and Ertepinar[8] have recently re-analyzed them and verified their solutions with experiments using silicone–rubber specimens[9].

As for problems involving finite torsional deformation of thick-walled shells, there has not been any stability analysis. In a recent account, Demiray and Suhubi[10] touched briefly the oscillational aspects of a finitely twisted solid cylinder. The purpose of the present work is to study both analytically and experimentally the torsional buckling phenomenon of long, cylindrical shells of arbitrary wall thickness. The material of the shell is assumed to be elastic, isotropic, homogeneous and incompressible.† More specifically, a neo-Hookean type of material is used in the formulation of the governing equations. The formulation is based on the rigorous theory of finite elasticity [1] and the extended theory of small displacements superposed on large elastic deformations[3]. Briefly, this method is the determination of the stress and the deformation fields of the elastic body under the external loads and then the superposing of a secondary, infinitesimal displacement field on the previously determined equilibrium state. The instability of the primary equilibrium state occurs when there exists a nontrivial secondary deformation field under the same external loading. The buckling torques and the associated buckling modes are determined from the solution of the governing field equations by means of the method of complementary functions in conjunction with the Runge–Kutta method of integration.

In the experiment, nine circular tubes of various thicknesses were made using a type of silicone rubber, which is very elastic and is virtually incompressible. These tubes were tested by means of a standard torsional apparatus. The test results verified the neo-Hookean assumption for the material and favorably verified the analytical results of the stability analysis.

FORMULATION OF THE PROBLEM

Consider a long cylindrical tube with a circular, hollow cross-section defined by an inner radius A_1 and an outer radius A_2. Torsion is applied so that planes, initially perpendicular to the axis of the tube, are rotated in their own plane through an angle of twist α proportional to the distance z of the plane from one end of the tube. Thus, a material point at coordinates (r, θ, z) in the twisted tube was originally at coordinates $(r, \theta-\alpha z, z)$. We assume that the material of the tube is perfectly elastic and incompressible. Its strain energy density function is denoted by $W(\mathrm{I}, \mathrm{II})$, where I and II are the first and the second strain invariants. The stress field induced by the prescribed deformation is given by the contravariant stress tensor[1].

$$\begin{aligned} &\tau^{11} = \alpha^2 \int_{A_1}^{r} \Phi r \,\mathrm{d}r & &\tau^{22} = r^{-2}\tau^{11} + \alpha^2\Phi \\ &\tau^{33} = \tau^{11} - \alpha^2\Psi r^2 & &\tau^{23} = (\Phi + \Psi)\alpha \\ &\tau^{12} = \tau^{13} = 0. \end{aligned} \tag{1}$$

where

$$\Phi = 2W_{,\mathrm{I}} \qquad \Psi = 2W_{,\mathrm{II}}. \tag{2}$$

In obtaining (1), it is assumed that the inner surface of the tube is stress free. However, on the outer surface of the tube a uniform normal pressure

$$q = \alpha^2 \int_{A_1}^{A_2} \phi r \,\mathrm{d}r \tag{3}$$

must be applied. Similarly, on the end faces of the tube, a normal force

$$N = 2\pi\alpha^2 \int_{A_1}^{A_2} \left[\int_{A_1}^{r} \phi r \,\mathrm{d}r - \psi r^2 \right] r \,\mathrm{d}r \tag{4}$$

†The assumption of material incompressibility is not a prerequisite as far as the method is concerned. It is used here primarily because the material used in the experiment was practically incompressible. For compressible hyperelastic materials, the development of the material constitutive equations is usually more complicated.

and an end torque

$$M = 2\pi\alpha \int_{A_1}^{A_2} (\phi + \psi) r^3 \, \mathrm{d}r \tag{5}$$

must also be applied.

It is seen from equation (3) that for the prescribed pure torsional deformation, the outer surface of the tube could not be made stress free. This is the well known secondary Poynting effect.† Similarly, the presence of the normal force N, equation (4), which is also secondary, is the result of the plane strain assumption.

We now consider an infinitesimal displacement field $w_i(r, \theta, z)$, $i = 1, 2, 3$; and superpose w_i onto the twisted tube. The induced, additional stress field is given by (see, e.g. [2]),

$$\begin{aligned} \hat{\tau}^{11} &= -2Pw_{1,r} + p', & \hat{\tau}^{22} &= -2Pr^{-3}w_1 - 2Pr^{-4}w_{2,\theta} + p'r^{-2}, \\ \hat{\tau}^{33} &= -2Pw_{3,z} + p', & \hat{\tau}^{12} &= -Pr^{-2}(w_{1,\theta} + w_{2,r} - 2r^{-1}w_2), \\ \hat{\tau}^{23} &= -Pr^{-2}(w_{2,z} + w_{3,\theta}), & \hat{\tau}^{31} &= -P(w_{1,z} + w_{3,r}). \end{aligned} \tag{6}$$

where $p' = p'(r, \theta, z)$ is an unknown pressure and

$$P = \Phi\left[\frac{\alpha^2}{2}(r^2 - A_1^2) - 1\right]. \tag{7}$$

In obtaining (6) and (7) we have assumed for definiteness, but without loss of generality, that the material of the tube is of neo-Hookean type.‡ Thus $W(\mathrm{I}, \mathrm{II}) = C_1(\mathrm{I}\text{-}3)$, with C_1 being the only material constant.

The equilibrium equations for the secondary state of deformation are

$$w_{1,rr} + (\alpha^2 + r^{-2})w_{1,\theta\theta} + w_{1,zz} + (r^{-1} - r\alpha^2)w_{1,r} + 2\alpha w_{1,\theta z} - (\alpha^2 + r^{-2})w_1 - 2\alpha r^{-1}w_{2,z} - 2(\alpha^2 r^{-1} + r^{-3})w_{2,\theta} + p'_{,r}/2C_1 = 0. \tag{8}$$

$$(2r^{-3} + \alpha^2 r^{-1})w_{1,\theta} + 2\alpha r^{-1}w_{1,z} + r^{-2}w_{2,rr} + (\alpha^2 r^{-2} + r^{-4})w_{2,\theta\theta} + r^{-2}w_{2,zz} + 2\alpha r^{-2}w_{2,\theta z} - r^{-3}w_{2,r} + p'_{,\theta}r^{-2}/2C_1 = 0. \tag{9}$$

$$-\alpha^2 r w_{1,z} + w_{3,rr} + (\alpha^2 + r^{-2})w_{3,\theta\theta} + 2\alpha w_{3,\theta z} + r^{-1}w_{3,r} + w_{3,zz} + p'_{,z}/2C_1 = 0. \tag{10}$$

The incompressibility condition associated with the secondary deformation yields

$$w_{1,r} + r^{-1}w_1 + r^{-2}w_{2,\theta} + w_{3,z} = 0. \tag{11}$$

Equations (8)–(11) govern the four unknown functions w_i and p' of the secondary deformation state of the tube. To determine whether or not the initially twisted state of the tube is unstable, we require that non-trivial solutions to the field equations (8)–(11) must satisfy the

†On the experiment, we have disregarded this effect, i.e. both the inner and the outer surfaces were free of tractions.
‡Additionally, a neo-Hookean type material was selected to insure the experimental verification of the analytical solution.

boundary conditions that the secondary surface tractions do not exist. Thus:

$$\begin{aligned} &w_{1,r} + p'/4C_1 = 0 \\ &w_{1,\theta} + w_{2,r} - 2r^{-1}w_2 = 0 \qquad r = A_1, A_2. \\ &w_{1,z} + w_{3,r} = 0. \end{aligned} \tag{12}$$

We now seek the solution of the linear system of equations (8)–(12) by proposing the solution form

$$\begin{aligned} w_1(r,\theta,z) &= i\sum_{n=-\infty}^{\infty} U_{1n}(r)\,\mathrm{e}^{i(n\theta+kz)} \\ w_2(r,\theta,r) &= \sum_{n=-\infty}^{\infty} U_{2n}(r)\,\mathrm{e}^{i(n\theta+kz)} \\ w_3(r,\theta,z) &= \sum_{n=-\infty}^{\infty} U_{3n}(r)\,\mathrm{e}^{i(n\theta+kz)} \\ p'(r,\theta,z) &= i\sum_{n=-\infty}^{\infty} U_{4n}(r)\,\mathrm{e}^{i(n\theta+kz)}. \end{aligned} \tag{13}$$

Substitution of (13) into (8)–(11) and a subsequent elimination of $U_{3n}(r)$ yield, for each integer n, three, second order ordinary differential equations in U_{1n}, U_{2n} and U_{4n} as functions of r only. For convenience, we express these equations without the subscript n,

$$\begin{aligned} &-(\alpha^2 r + r^{-1})U_1'' - (2\alpha^2 + r^{-2})U_1' + [\alpha^2k^2r + r^{-3}(3n^2+1) + r^{-1}(\alpha n + k)^2]U_1 - r^{-2}nU_2'' \\ &+ [nr^{-3} - 2\alpha r^{-1}(\alpha n + k)]U_2' + [2\alpha r^{-2}(\alpha n + k) + r^{-4}(n^3 + 2n) + r^{-2}n(\alpha n + k)^2]U_2 \\ &\qquad + U_4''/2C_1 - k^2U_4/2C_1 = 0. \end{aligned} \tag{14}$$

$$\begin{aligned} &U_1'' + (r^{-1} - \alpha^2 r)U_1' - [\alpha^2 + r^{-2}(n^2+1) + (\alpha n + k)^2]U_1 \\ &\qquad - [2nr^{-3} + r^{-1}(2n\alpha^2 + 2\alpha k)]U_2 + U_4'/2C_1 = 0. \end{aligned} \tag{15}$$

$$U_2'' - r^{-1}U_2' - [n^2r^{-2} + (\alpha n + k)^2]U_2 - [2nr^{-1} + \alpha^2nr + 2\alpha kr]U_1 - nU_4/2C_1 = 0. \tag{16}$$

Similarly, the boundary conditions (12) reduce to

$$\begin{aligned} &U_1' + U_4/4C_1 = 0 \\ &nU_1 - U_2' + 2r^{-1}U_2 = 0 \qquad r = A_1, A_2 \\ &\alpha^2rU_1' + [\alpha^2 + k^2 + n^2r^{-2} + (\alpha n + k)^2]U_1 + nr^{-2}U_2' + 2\alpha r^{-1}(\alpha n + k)U_2 - U_4'/2C_1 = 0. \end{aligned} \tag{17}$$

In order to determine a set of non-trivial solutions for U_1, U_2 and U_4 satisfying (14)–(17), the characteristic determinant of the system must vanish for each given set of proper values of α, n and k. It is noted that α represents the prestressed state of the tube, while n and k define, respectively, the circumferential and the axial modes of deformation in the secondary state of the tube. In the numerical computations, we choose to specify the modal numbers n and k and determine the smallest α for which the characteristic determinant vanishes. This critical value of α yields through equation (5) the critical end torque M under which the tube will buckle in the mode corresponding to n and k.

METHODS OF SOLUTION

A numerical scheme will be adopted here to solve the system of equations (14)–(17). Briefly, we first non-dimensionalize the system by introducing the following quantities;

$$\bar{\alpha} = \alpha A_1, \qquad \bar{k} = kA_1, \qquad y = r/A_1$$
$$f_4 = U_1/A_1, \qquad f_5 = U_2/A_1^2, \qquad f_6 = U_4/2C_1. \tag{18}$$

Secondly, we transform the three second order equations (14)–(16) to six first order equations in the form

$$f_i'(y) = F_i(f_m, y) \qquad i, m = 1, 2, \ldots 6. \tag{19}$$

where

$$F_1 = (\bar{\alpha}^2 y - y^{-1})f_1 - f_3 + [\bar{\alpha}^2 + y^{-2}(n^2+1) + (\bar{\alpha}n + \bar{k})^2]f_4 + [2ny^{-3} + 2\bar{\alpha}y^{-1}(\bar{\alpha}n + \bar{k})]f_5. \tag{20}$$

$$F_2 = y^{-1}f_2 + [2ny^{-1} + \bar{\alpha}^2 ny + 2\bar{\alpha}\bar{k}y]f_4 + [n^2y^{-2} + (\bar{\alpha}n + \bar{k})^2]f_5 + nf_6. \tag{21}$$

$$F_3 = \bar{\alpha}^2(\bar{\alpha}^2y^2 + 2)f_1 + 2\bar{\alpha}y^{-1}(\bar{\alpha}n + \bar{k})f_2 - (\bar{\alpha}^2 y + y^{-1})f_3 + [\bar{\alpha}^2(1+n^2)(\bar{\alpha}^2 y + 2y^{-1})$$
$$+ 2\bar{\alpha}n\bar{k}(\bar{\alpha}^2 y + y^{-1})]f_4 + [2\bar{\alpha}^2(\bar{\alpha}\bar{k} + n\bar{\alpha}^2 + ny^{-2})]f_5 + (\bar{k}^2 + n^2y^{-2})f_6 \tag{22}$$

and

$$F_4 = f_1 = f_4', \qquad F_5 = f_2 = f_5', \qquad F_6 = f_3 = f_6'. \tag{23}$$

Similarly, the boundary conditions in (17) reduce to

$$\begin{aligned} &f_1 + \tfrac{1}{2}f_6 = 0. \\ &nf_4 - f_2 + 2y^{-1}f_5 = 0. \qquad y = 1, A_2/A_1 \\ &\bar{\alpha}^2 y f_1 + [\bar{\alpha}^2 + \bar{k}^2 + n^2y^{-2} + (\bar{\alpha}n + \bar{k})^2]f_4 + ny^{-2}f_2 + 2\bar{\alpha}y^{-1}(\bar{\alpha}n + \bar{k})f_5 - f_3 = 0. \end{aligned} \tag{24}$$

The system of six first order equations (19) are now solved by the so-called complementary functions method[11, 12]. Essentially, the method reduces the two-point boundary value problem into an initial value problem and uses a numerical scheme of integration. In this case, the integration procedure follows that of the Runge–Kutta method of order two. Thus, the six equations in (19) are integrated numerically six times to obtain six sets of solutions $f_i^{(j)}$ $j = 1, 2, \ldots 6$, the jth set of solutions $_i^{(j)}$ satisfying the initial conditions

$$f_i^{(j)} = \delta_{ij} \qquad i,j = 1, 2, \ldots 6. \quad \text{at} \quad y = 1 \tag{25}$$

where δ_{ij} is the Kronecker delta function.

Clearly, a linear combination of the solutions $f_i^{(j)}$ in the form

$$f_i = B_j f_i^{(j)} \qquad i, j = 1, 2, \ldots 6 \tag{26}$$

is also a solution to the system of equations (19). The six constants B_j are determined from the conditions that (26) must also satisfy the six homogeneous boundary conditions (24). Thus, in order to have non-trivial solutions for (26), satisfaction of (24) leads to the vanishing of a six by six determinant containing now only the following parameters, namely $\bar{\alpha}$, $\bar{k}$, A_1/A_2 and n. Thus, for a given shell A_1/A_2 is known. If we also specify the values of n and $\bar{k}$, the vanishing characteristic determinant reduces to a transcendental equation in $\bar{\alpha}$. Clearly, the buckling of the twisted tube is characterized by the set of values of n and $\bar{k}$ for which the first root $\bar{\alpha}$ is the smallest. This value of $\bar{\alpha}$ is the critical angle of twist and is denoted by $\bar{\alpha}_{cr}$. The critical end-torque may be computed from $\bar{\alpha}_{cr}$ through equations (18) and (5).

In the numerical computations, it was observed that if we let $\bar{k} \geq 0$ the transcendental equation will yield real roots only when $n < 0$. For $n = -1$, there existed no root for $\bar{\alpha}$ when $\bar{k}$ is larger than zero. However, as $\bar{k}$ approaches zero, the transcendental equation vanishes numerically for all real $\bar{\alpha}$. The latter has been discussed by Flügge[13] in his book on thin shells. In Ref. [13], it is shown that for the mode corresponding to $n = -1$ and $\bar{k} = 0$, every cross-section of the shell rotates about its diametrical axis, and the axis of the shell is deformed into a steep helical curve while the circular cross-sections remain normal to the deformed axis. Green and Spencer[5] have obtained similar results for a solid cylinder. In the present study, we shall exclude the case for $n = -1$ as it is irrelevant to the physical problem. Indeed, the numerical results indicate that, for all thickness ratios $0 < A_1/A_2 < 1$, $n = -2$ corresponds to the fundamental circumferential mode of buckling provided that the axial mode number $\bar{k}$ is properly chosen.

As for the value of $\bar{k}$ associated with the fundamental mode of instability, it is found that $\bar{k}$ depends upon the thickness ratio A_1/A_2. Again, Flügge[13] has shown for thin shells that the lowest axial mode of instability occurs when

$$\bar{k}^4 = \frac{4}{1-\sigma^2}(t/A)^2 \tag{27}$$

where σ is the Poisson ratio of material, t is the thickness of the shell and A is the mean radius. For shells of finite thickness, t/A may be expressed as

$$\frac{t}{A} = 2\left(\frac{1 - A_1/A_2}{1 + A_1/A_2}\right). \tag{28}$$

By taking $\sigma = 0{\cdot}5$ for incompressible materials, Flügge's results for the buckling torque corresponding to $\bar{k}_{cr}$ and $n = -2$ is given by (see Ref. [13], p. 438)

$$\bar{M}_{cr} = \frac{3M_{cr}}{EA_1^3} = 31\left(\frac{A_2 - A_1}{A_2 + A_1}\right)^{5/2}. \tag{29}$$

A comparison of the present numerical results against Flügge's shows favorable agreement for thin, and even moderately thick shells. Table 1 shows the values of $\bar{k}_{cr}$ found in the present study compared to Flügge's, found from equation (27). There is close agreement for shells with thickness ratios between 1·0 and 0·8. The two sets of results diverge rapidly, however, as the shell thickness becomes larger.

In Fig. 1, the critical end torque M_{cr} as found in the present study and as evaluated numerically from equation (29) is plotted against the A_1/A_2 ratio. Again, Flügge's results agree with the

Table 1. Comparison of critical $\bar{k}$

A_1/A_2	0·95	0·9	0·8	0·7	0·6	0·5	0·4	0·35	0·3	0·2
$\bar{k}$, present	0·35	0·5	0·7	0·8	0·9	1·05	1·0	1·0	0·9	0·7
$\bar{k}$, Flügge	0·34	0·49	0·71	0·90	1·07	1·24	1·40	1·49	1·57	1·76

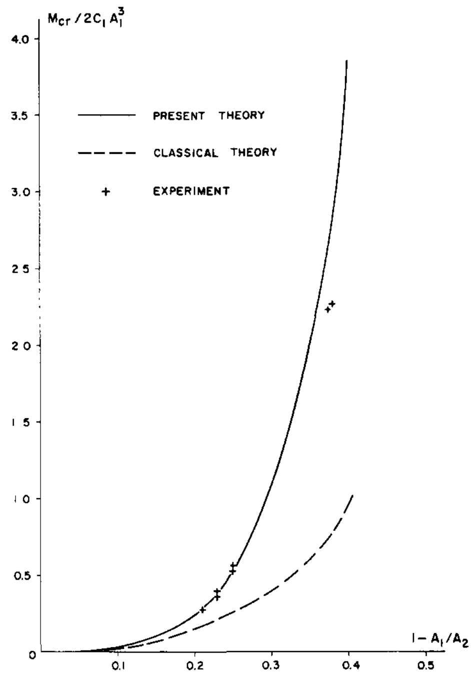

Fig. 1. Critical buckling torque vs thickness of tube.

present results only for thin or moderately thick shells, but the thin shell theory always underestimates the torsional buckling load.

Finally, in examining the convergence of the solution scheme, we have undertaken two independent checking procedures. First, in applying the Runge–Kutta integration scheme we used from 20 to 100 pivotal points throughout the thickness of the tube. The improvement of the results (in this case, the value of $\bar{\alpha}_{cr}$) from increasing the pivotal points from 20 to 100 was less than 5 per cent, even for very thick shells. Hence, in all subsequent computations only 20 pivotals were taken. Secondly, we have also programmed a solution scheme using the method of finite-differences with a central-difference scheme for the field equations and a forward and a backward difference scheme for the two end points respectively. We have obtained essentially the same results from the two methods of solution, although the complementary functions method is much more advantageous in that it is easier to program, needs less storage, needs less computing time for the same number of pivotals, and more significantly, it converges much faster and thus requires a lesser number of pivotals than the finite difference scheme.

EXPERIMENT

We have manufactured nine silicone rubber cylindrical tubes for the purpose of experiment. Techniques involved in making these specimens were reported earlier in Ref. [9]. Table 2 shows

Table 2. Geometry and material constants of specimens

Tube	Inner radius (in.)	A_1/A_2	Length (in.)	C_1 (psi)	M_{cr} (in-lbs)
1	0·625	0·77	10	38	7·5
2	0·625	0·77	10	38	6·7
3	0·75	0·79	10	40	9·3
4	0·75	0·75	10	40	18·5
5	0·75	0·75	10	37	17·5
6	0·50	0·62	10	36	20·1
7	0·50	0·61	10	37·5	21·5
8	0·50	0·50	10	36	80†
9	0·50	0·50	10	38·4	81†

†Tube did not buckle under twist due to premature failure of the end clamps.

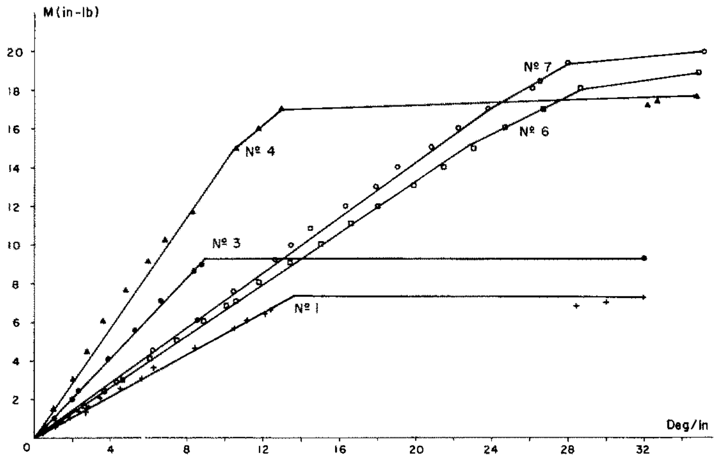

Fig. 2. Applied torque vs the angle of twist of tube.

the various geometrical and material parameters of these tubes. The tubes were tested on a standard torsional machine, which affixes each end of the tube by means of a clamping device. One end of the tube is rotated while the other end is held stationary. The applied torque and the corresponding angle of twist may be read directly from the machine. Figure 2 shows the torque vs angle of twist relationships for tubes No. 1, 3, 4, 6 and 7. The $M-\alpha$ curves are essentially linear prior to torsional buckling, as is expected for a neo-Hookean type of material.† The slope of the linear portion of the curves gives the value of the constant C_1 through solutions of equations (2) and (5). When the applied torque reaches its critical value for a given shell, the angle of twist of the shell increases suddenly as is depicted in Fig. 2. When this occurs, the shell is twisted in the form having a helical surface. It was observed that the buckled shape of the shell compared well with the predicted values for n and k. The precise experimental critical torque M_{cr} is determined in accordance with the well-known Southwell plot technique (see, e.g. Ref. [9]). It is seen from Fig. 1 that the experimental results agree closely with the theoretical prediction. The two tubes with $A_1/A_2 = 0{\cdot}5$ could not be tested to their buckled state because of a premature failure of the

†The same material was earlier studied experimentally under uniaxial tension and triaxial compression; see Ref. [9].

clamping device. For these two tubes, the predicted critical torque was 109 in-lb, while the actual maximum torque that could be applied before the clamping device failed was 80 in-lb.

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REFERENCES

1. A. E. Green and W. Zerna, *Theoretical Elasticity* 2nd edition. Oxford University Press (1968).
2. A. E. Green and A. Adkins, *Large Elastic Deformations and Non-Linear Continuum Mechanics.* Oxford University Press (1960).
3. A. E. Green, R. S. Rivlin and R. T. Shield, General Theory of Small Elastic Deformation Superposed on Finite Elastic Deformations, *Proc. Roy. Soc.* **A211** (1952).
4. E. W. Wilkes, On the Stability of a Circular Tube under End Thrust, *Quart. J. Mech. Appl. Math.* **8** (1955).
5. A. E. Green and A. J. M. Spencer, The Stability of Circular Cylinder Under Finite Extension and Torsion, *J. Math. Phys.* **37** (1958).
6. Z. Wesolowski, Stability of an Elastic Thick-Walled Spherical Shell Loaded by an External Pressure, *Arch. Mech. Stosow.* **19** (1967).
7. J. L. Nowinski and M. Shahinpoor, Stability of an Elastic Circular Tube of Arbitrary Wall Thickness Subjected to an External Pressure, *Int. J. Non-Lin. Mech.* **4** (1969).
8. A. S. D. Wang and A. Ertepinar, Stability and Vibrations of Elastic Thick-walled Cylindrical and Spherical Shells Subjected to Pressure, *Int. J. Non-Lin. Mech.* **7** (1972).
9. A. S. D. Wang and A. Ertepinar, Experiment on the Sudden Collapse of Thick-walled Cylinders Undergoing Large Elastic Deformations, *10th Anniv. Vol. Soc. Eng. Sci.*, North Carolina Univ. (1974).
10. H. Demiray and E. Suhubi, Small Torsional Oscillations of an Initially Twisted Rubber Cylinder, *Int. J. Eng. Sci.* **8** (1970).
11. T. R. Goodman and G. N. Lance, The Numerical Integration of Two-Point Boundary Value Problems, *Math. Table and Other Aids to Computations*, **10, No. 54** (1956).
12. G. N. Lance, *Numerical Methods for High-speed Computers.* Iliffe and Sons, London (1960).
13. W. Flügge, *Stresses in Shells.* Springer (1967).